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Linear maps and additive maps that preserve operators annihilated by a polynomial ☆

Zhaofang Bai [a,b] and Jinchuan Hou [b,c,*]

[a] *School of Science, Xi'an Jiaotong University, Xi'an, 710049, PR China*
[b] *Department of Mathematics, Shanxi Teachers University, Linfen, 041004, PR China*
[c] *Department of Mathematics, Shanxi University, Taiyuan, 030000, PR China*

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Abstract

Let H, K be infinite-dimensional complex Hilbert spaces. And let $p(t)$ be a complex polynomial with $\deg(p) \geqslant 2$. In this paper we give a sufficient and necessary condition for a surjective linear or additive map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ to preserve operators annihilated by p in both directions and answer a question raised by Šemrl affirmatively. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

The problem of characterizing linear operators on matrix algebras that leave invariant certain functions, subsets or relations has attracted the attention of many mathematicians in the last few decades (see the survey paper [1]). In fact, it presents one of the most active areas in matrix theory. The first papers concerning this problem [2,3] date back to the 19th century. It seems that the systematic

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* Corresponding author.
E-mail address: jhou@dns.sxtu.edu.cn (J. Hou).

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study of linear preservers begin with the paper of Marcus and Moyls [4]. They characterized linear maps on M_n, the algebra of all $n \times n$ matrices, that preserve the spectrum. Recently, an interesting generalization of this result was obtained. Li and Pierce got the general form of bijective linear operators on M_n mapping the set of matrices annihilated by a given polynomial into itself [5], thus extending not only the above-mentioned results due to Marcus and Moyls but also several results on linear mappings preserving nilpotents, idempotents, or r-potents. In recent years interests in similar questions on operator algebras over infinite-dimensional spaces have also been growing (for example, [6–9] and references therein). Šemrl continues Li and Pierce's work by studying linear maps $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ that preserve operators annihilated by a given polynomial [8]. Here H is a complex Hilbert space and $\mathcal{B}(H)$ is the algebra of all bounded linear operators on H. These results can be considered as extensions of a well-known theorem due to Eidelheit [10] which states that all algebraic automorphisms of $\mathcal{B}(X)$ are inner; here X is a Banach space.

When discussing an automorphism of an operator algebra on Hilbert space one usually assumes that this map is linear. A more general approach would be to consider an algebra as only a ring. Let us recall that a ring automorphism of an algebra is a bijective additive and multiplicative map. So it is not assumed to be linear. An interesting result concerning ring automorphisms has been obtained by Arnold [11]: Every ring automorphism of the algebra $\mathcal{B}(X)$ of all bounded operators on an infinite-dimensional Banach space X is automatically real-linear (or alternatively, it is either linear or conjugate-linear relative to complex scalars). It seems natural to study not only linear preservers, but also additive ones. The first step in this direction appears to be a generalization of the Jafarian–Sourour result to that of additive spectrum-preserving maps [12]. In this direction, other interesting results have been gotten (for example, [13–15]). These results motivate us to continue the study of additive maps on operator algebras.

Let H, K be infinite-dimensional complex Hilbert spaces. In [8], Šemrl proved: if $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a surjective linear map satisfying $\Phi(I) = I$ and preserves square zero in both directions (that is, $A^2 = 0 \Leftrightarrow \Phi(A)^2 = 0$), then Φ is either an automorphism or an anti-automorphism. There he asked two questions which are still open. One is whether we can remove the assumption $\Phi(I) = I$, the other is whether the same result still holds true under a weaker assumption of preserving square zero in one direction only.

One of the purposes of our present paper is to give an affirmative answer to Šemrl's first question and then to classify the surjective linear maps and additive maps on $\mathcal{B}(H)$ which preserve the subset annihilated by a polynomial. The paper is organized as follows.

Section 2 is about linear maps which preserve operators annihilated by a polynomial. We show that if $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is bounded and preserves square zero operators (in one direction only), or if Φ preserves square zero operators in both directions, then Φ is either an isomorphism or an anti-automorphism multiplied by a scalar (Theorem 2.3). These give affirmative answers to Šemrl's questions, especially to the first one. Based on it, we obtain a characterization of a linear map Φ which preserves operators annihilated by a polynomial in both directions without the assumption that Φ is unital (Theorem 2.4), which generalizes the main result in [8].

In Section 3 we study the surjective additive maps which preserve operators annihilated by a polynomial in both directions and give a complete classification of them, again, without the assumption that Φ is unital. Let p be a polynomial. Denote by $\mathcal{Z}(p)$ the set of all zeros of p and by $\mathcal{G}(p)$ the set $\{\lambda \in \mathbb{C}\backslash\{0\}$: $\lambda\mathcal{Z}(p) \subseteq \mathcal{Z}(p)\}$. A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ preserves the operators annihilated by p if $p(\Phi(T)) = 0$ whenever $p(T) = 0$. We show that if $\deg(p) \geqslant 2$ and if $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a surjective additive map satisfying $\Phi(\mathbb{C}P) \subset \mathbb{C}\Phi(P)$ for every rank-1 idempotent P, then Φ preserves operators annihilated by $p(t)$ in both directions if and only if there exists a complex number $c \in \mathcal{G}(p)$ and an invertible bounded linear or conjugate-linear operator $A : H \to K$ such that either $\Phi(T) = cATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = cAT^*A^{-1}$ for all $T \in \mathcal{B}(H)$. Moreover, if $\deg(p) > 2$, the condition $\Phi(\mathbb{C}P) \subset \mathbb{C}\Phi(P)$ can be removed (Theorem 3.8).

Now we fix some notations. Let H, K be infinite-dimensional complex Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$, and $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ a linear or an additive map. We say Φ preserves operators annihilated by a polynomial $p(t)$ (in both directions) if for every $T \in \mathcal{B}(H)$, $p(\Phi(T)) = 0$ whenever $p(T) = 0$ (if and only if $p(T) = 0$); Φ preserves square zero operators (in both directions) if $\Phi(T)^2 = 0$ whenever $T^2 = 0$ (if and only if $T^2 = 0$); Φ preserves k-nilpotent operators (in both directions) if $\Phi(T)^k = 0$ whenever $T^k = 0$ (if and only if $T^k = 0$). We denote $\dim(\operatorname{rng}(P))$ by $\dim P$, where $\operatorname{rng}(P)$ is the range of an idempotent operator P. Throughout this paper, we will denote by $x \otimes y$ the bounded linear operator on H defined for any $x, y \in H$ by $(x \otimes y)(z) = \langle z, y\rangle x$ for arbitrary $z \in H$. Note that this operator is of rank one whenever x and y nonzero, and that every operator of rank one can be written in this form with x, y are nonzero. Recall also that every operator of finite rank can be expressed as a sum of operators of rank one. By a projection we mean a bounded self-adjoint idempotent operator. Two idempotent operators P_1, P_2 are said to be orthogonal if $P_1P_2 = P_2P_1 = 0$. By $\mathcal{F}(H)$, $\mathcal{N}(H)$, $\mathcal{N}_1(H)$, and $\mathcal{N}^k(H)$ we denote the set of all finite rank linear bounded operators, the set of all nilpotent bounded linear operators, the set of all nilpotent bounded linear operators of rank 1, and the set of all nilpotent bounded linear operators with nilindex not greater than k, i.e., the set of all k-nilpotent operators.

2. Linear maps preserving operators annihilated by a polynomial

It was proved by Šemrl in [8] that if Φ is a surjective linear map satisfying $\Phi(I) = I$, and if Φ preserves square zero operators in both directions, then Φ is either an automorphism or an anti-automorphism of $\mathcal{B}(H)$. Furthermore, Šemrl asked:

Question 1. Can the condition $\Phi(I)=I$ in above result be omitted?

Question 2. Can the assumption "in both directions" in above result be omitted?

This section is devoted to giving some characterizations of square-zero-preserving linear maps and solving these questions. We also obtain a characterization of surjective linear maps preserving operators annihilated by a polynomial $p(t)$ in both directions, which generalizes the main theorem in [8].

We begin with a lemma which will play an important role in the sequel.

Lemma 2.1. *Suppose that $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is a surjective additive map which preserves square-zero operators and $P_1, P_2\in\mathcal{B}(H)$ are two orthogonal idempotents. Then*

$$\Phi(P_1+P_2)^2=\Phi(P_1)^2+\Phi(P_2)^2. \tag{2.1}$$

Proof. We have the following five cases to check:

(i) $\dim P_i=\dim(I-P_i)=\infty$ for $i=1$ or 2;
(ii) $\dim P_1<\infty$, $\dim P_2<\infty$;
(iii) $\dim P_1<\infty$, $\dim(I-P_2)<\infty$;
(iv) $\dim(I-P_1)<\infty$, $\dim P_2<\infty$;
(v) $\dim(I-P_1)<\infty$, $\dim(I-P_2)<\infty$.

In case (i), without loss of generality, we assume that $\dim P_1=\dim(I-P_1)=\infty$. Let $A,B\in\mathcal{B}(H)$ satisfy $P_1AP_1=A$ and $(I-P_1)B(I-P_1)=B$. It follows from [16, Theorem 2] that A and B can be written as the sum of five operators with square zero. Say $A=A_1+A_2+A_3+A_4+A_5$ and $B=B_1+B_2+B_3+B_4+B_5$, with $P_1A_iP_1=A_i$ and $(I-P_1)B_i(I-P_1)=B_i$ $(i=1,\dots,5)$. Clearly, $(A_i+B_j)^2=0$. Consequently, we have $\Phi(A_i)\Phi(B_j)+\Phi(B_j)\Phi(A_i)=0$, which further yields

$$\Phi(A)\Phi(B)+\Phi(B)\Phi(A)=0.$$

With $A=P_1$, $B=P_2$, we get

$$\Phi(P_1+P_2)^2=\Phi(P_1)^2+\Phi(P_2)^2.$$

In case (ii), we can find P_3 such that it is orthogonal to P_1+P_2 and $\dim P_3=\dim(I-P_3)=\infty$. Thus

$$\begin{aligned}\Phi(P_1+P_2)^2+\Phi(P_3)^2&=\Phi(P_1+P_2+P_3)^2=\Phi(P_1)^2+\Phi(P_2+P_3)^2\\&=\Phi(P_1)^2+\Phi(P_2)^2+\Phi(P_3)^2,\end{aligned}$$

so $\Phi(P_1+P_2)^2=\Phi(P_1)^2+\Phi(P_2)^2$. Similarly, we can show that Eq. (2.1) holds true for the remaining three cases, too. □

Corollary 2.2. *Suppose that $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is a surjective additive map which preserves square-zero operators. Then*

$$\Phi(R)\Phi(I)+\Phi(I)\Phi(R)=2\Phi(R)^2, \tag{2.2}$$

$$\Phi(I)^2\Phi(R)=\Phi(R)\Phi(I)^2 \quad \text{and} \quad \Phi(R)^2\Phi(I)=\Phi(I)\Phi(R)^2 \tag{2.3}$$

hold for all idempotents $R\in\mathcal{B}(H)$.

Proof. It follows immediately from Lemma 2.1 that $\Phi(R)\Phi(I-R)+\Phi(I-R)\Phi(R)=0$ which implies that Eq. (2.2) is true.

Thus we have

$$\Phi(R)^2\Phi(I)+\Phi(R)\Phi(I)\Phi(R)=2\Phi(R)^3$$

and

$$\Phi(I)\Phi(R)^2+\Phi(R)\Phi(I)\Phi(R)=2\Phi(R)^3,$$

which imply that

$$\Phi(R)^2\Phi(I)=\Phi(I)\Phi(R)^2$$

holds for every idempotent R. Consequently, it follows from

$$\Phi(I)^2\Phi(R)+\Phi(I)\Phi(R)\Phi(I)=2\Phi(I)\Phi(R)^2$$

and

$$\Phi(R)\Phi(I)^2+\Phi(I)\Phi(R)\Phi(I)=2\Phi(R)^2\Phi(I)$$

that

$$\Phi(R)\Phi(I)^2=\Phi(I)^2\Phi(R) \quad \text{for every idempotent } R.$$

So, (2.3) is true and the proof is finished. □

The following theorem omits the assumption that $\Phi(I)=I$ in the Šemrl's result [8] and gives an answer to Question 1 and partially Question 2 affirmatively.

Theorem 2.3. *Let $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ be a surjective linear map. Then Φ preserves square zero in both directions* (*or, Φ is bounded and preserves square zero*) *if and only if there exists a nonzero complex number c and an invertible operator $A\in\mathcal{B}(H,K)$ such that either $\Phi(T)=cATA^{-1}$ for all $T\in\mathcal{B}(H)$ or $\Phi(T)=cAT^{\mathrm{tr}}A^{-1}$ for all $T\in\mathcal{B}(H)$, where T^{tr} denotes the transpose of T relative to a fixed but arbitrary orthonormal base of H.*

Proof. We need only prove the "only if" part. Suppose Φ preserves square zero in both directions. First we prove that Φ is injective. Assume to the contrary that there exists a nonzero $A \in \mathcal{B}(H)$ such that $\Phi(A) = 0$. Then A is a square zero operator and it is easy to find a square zero operator B such that $A + B$ is not a square zero operator. It follows that $\Phi(B) = \Phi(A + B)$ is not a square zero operator. This contradiction shows that Φ must be injective.

It is known from [16] that every element in $\mathcal{B}(H)$ can be written as a sum of at most 5 idempotents. Thus by Corollary 2.2, $\Phi(I)^2\Phi(A) = \Phi(A)\Phi(I)^2$ holds for every $A \in \mathcal{B}(H)$. The bijectivity of Φ tells us that $\Phi(I)^2 = \mu I$ for some nonzero complex number μ. Without loss of generality, assume $\Phi(I)^2 = I$. Then there is an idempotent P_1 such that $\Phi(I) = 2P_1 - I$. Applying above argument to Φ^{-1}, we obtain that $\Phi^{-1}(I)^2 = \delta I$ for some nonzero $\delta \in \mathbb{C}$, and $\Phi^{-1}(I)\Phi^{-1}(E) + \Phi^{-1}(E)\Phi^{-1}(I) = 2\Phi^{-1}(E)^2$, $\Phi^{-1}(E)^2\Phi^{-1}(I) = \Phi^{-1}(I)\Phi^{-1}(E)^2$ hold for all idempotents $E \in \mathcal{B}(K)$. Since $\Phi(I) = 2P_1 - I$, one has $\Phi^{-1}(I) = 2\Phi^{-1}(P_1) - I$, and

$$\bigl(2\Phi^{-1}(P_1) - I\bigr)\Phi^{-1}(E) + \Phi^{-1}(E)\bigl(2\Phi^{-1}(P_1) - I\bigr) = 2\Phi^{-1}(E)^2$$

holds for all idempotents E. Choose $E = P_1$; we get

$$2\Phi^{-1}(P_1)^2 - \Phi^{-1}(P_1) + 2\Phi^{-1}(P_1)^2 - \Phi^{-1}(P_1) = 2\Phi^{-1}(P_1)^2.$$

So

$$\Phi^{-1}(P_1) = \Phi^{-1}(P_1)^2;$$

i.e., $P_2 = \Phi^{-1}(P_1)$ is an idempotent. Now it is clear that $\Phi^{-1}(I)^2 = I$.

On the other hand, from the identities $\Phi(R)\Phi(I) + \Phi(I)\Phi(R) = 2\Phi(R)^2$ and $\Phi(I) = 2P_1 - I$ we have

$$(2P_1 - 1)\Phi(R) + \Phi(R)(2P_1 - 1) = 2\Phi(R)^2,$$
$$P_1\Phi(R) + \Phi(R)P_1 = \Phi(R)^2 + \Phi(R).$$

It follows that

$$\bigl(P_1 - \Phi(R)\bigr)^2 = P_1 - \Phi(R),$$

and hence

$$\Phi(P_2 - R)^2 = \Phi(P_2 - R)$$

holds for all idempotents R. Let $T \in P_2\mathcal{B}(H)(I - P_2)$ be arbitrary; then $(P_2 + T)^2 = P_2 + T$ and $T^2 = 0$. Thus

$$0 = \Phi(T)^2 = \Phi\bigl(P_2 - (P_2 + T)\bigr)^2 = \Phi\bigl(P_2 - (P_2 + T)\bigr) = -\Phi(T).$$

So $T = 0$. This implies that $P_2\mathcal{B}(H)(I - P_2) = \{0\}$. Hence we must have $P_2 = 0$ or $P_2 = I$. Now it is clear that $\Phi^{-1}(I) = \pm I$ and $\Phi(I) = \pm I$.

Thus we have proved by now that $\Phi(I) = cI$ for some nonzero scalar c. Without loss of generality we may assume that $\Phi(I) = I$. Applying $\Phi(R)\Phi(I) +$

$\Phi(I)\Phi(R) = 2\Phi(R)^2$ again, we know that Φ preserves idempotents in both directions. Using directly [6, Theorem 1], one completes the proof. □

For a complex polynomial $p(t)$, let $\mathcal{Z}(p)$ be the set of all zeros of p and let $\mathcal{G}(p) = \{\lambda \in \mathbb{C}\backslash\{0\} \mid \lambda\mathcal{Z}(p) \subseteq \mathcal{Z}(p)\}$. It is easily seen that if $\mathcal{Z}(p) \neq \{0\}$, then $\mathcal{G}(p)$ is a finite multiplicative subgroup of the unit circle, and thus, for some integer k, $\mathcal{G}(p) = \{\lambda \in \mathbb{C} \mid \lambda^k = 1\}$. If $\mathcal{Z}(p) = \{0\}$, i.e., if $p(t) = t^n$, then $\mathcal{G}(p) = \mathbb{C}\backslash\{0\}$.

Now Theorem 2.3 allows us to give a generalization of the main theorem in [8] by omitting the assumption $\Phi(I) = I$.

Theorem 2.4. *Let H and K be infinite-dimensional complex Hilbert spaces, and let $p(t)$ be a complex polynomial with* $\deg(p) \geqslant 2$. *Then a surjective linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ preserves operators annihilated by $p(t)$ in both directions if and only if there exists a complex number $c \in \mathcal{G}(p)$ and an invertible operator $A \in \mathcal{B}(H, K)$ such that either $\Phi(T) = cATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = cAT^{\mathrm{tr}}A^{-1}$ for all $T \in \mathcal{B}(H)$, where T^{tr} denotes the transpose of T relative to a fixed but arbitrary orthonormal base of H.*

Proof. Because the assumption $\Phi(I) = I$ in the main theorem of [8] is only needed when $p(t) = t^2$, it follows from Theorem 2.3 that if Φ preserves operators annihilated by $p(t)$ in both directions, then Φ is either an isomorphism or an anti-isomorphism multiplied by a scalar c. It is clear that $c \in \mathcal{G}(p)$. The inverse is obvious. □

3. Additive maps preserving operators annihilated by a polynomial

The main purpose of this section is to give a classification of surjective additive maps which preserve operators annihilated by a polynomial. We first discuss additive maps which preserve square zero operators.

Theorem 3.1. *Let H and K be infinite-dimensional complex Hilbert spaces and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a surjective additive map such that $\Phi(\mathbb{C}P) \subset \mathbb{C}\Phi(P)$ holds for every rank-*1 *idempotent P. Then Φ preserves square zero in both directions if and only if there exists a nonzero scalar c and a bounded invertible linear or conjugate-linear operator A from H onto K such that either $\Phi(T) = cATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = cAT^*A^{-1}$ for all $T \in \mathcal{B}(H)$.*

Proof. We divide the proof into several steps.

Step 1. Φ is injective and $\Phi(I) = cI$ for some nonzero complex number c. So, without loss of generality we may assume that $\Phi(I) = I$, and then Φ is idempotent preserving.

The proof is the same as that of Theorem 2.3, we omit it.

Step 2. For every idempotent $P \in \mathcal{B}(H)$, we have

$$\Phi\big(P\mathcal{B}(H)P\big) = \Phi(P)\mathcal{B}(K)\Phi(P)$$

and

$$\Phi\big((I-P)\mathcal{B}(H)(I-P)\big) = \big(I-\Phi(P)\big)\mathcal{B}(K)\big(I-\Phi(P)\big).$$

Let $X_1 = P\mathcal{B}(H)P$, $X_2 = P\mathcal{B}(H)(I-P)$, $X_3 = (I-P)\mathcal{B}(H)P$ and $X_4 = (I-P)\mathcal{B}(H)(I-P)$; and let $Y_1 = \Phi(P)\mathcal{B}(K)\Phi(P)$, $Y_2 = \Phi(P)\mathcal{B}(K)(I-\Phi(P))$, $Y_3 = (I-\Phi(P))\mathcal{B}(K)\Phi(P)$ and $Y_4 = (I-\Phi(P))\mathcal{B}(K)(I-\Phi(P))$. For arbitrary $T \in X_{2,}$ we have $(P+T)^2 = P+T$ and $T^2 = 0$. Thus $\Phi(T)^2 = 0$ and $\Phi(P+T)^2 = \Phi(P+T)$. It follows that

$$\Phi(T)\Phi(P) + \Phi(P)\Phi(T) = \Phi(T)$$

and

$$\big(I-\Phi(P)\big)\Phi(T)\Phi(P) + \Phi(P)\Phi(T)\big(I-\Phi(P)\big) = \Phi(T)$$

since $\Phi(P)\Phi(T)\Phi(P) + \Phi(P)\Phi(T) = \Phi(P)\Phi(T)$. Therefore, $\Phi(T) \in Y_2 + Y_3$ and $\Phi(X_2) \subset Y_2 + Y_3$. Similarly, we have $\Phi(X_3) \subset Y_2 + Y_3$. Now we prove $\Phi(X_1) \subset Y_1$. If P is of finite rank, for any $T \in X_1$ we have $T = \sum_{i=1}^{n} \lambda_i P_i$, for some finite set $\{P_i\}$ of rank-1 idempotents in X_1; if P is not of finite rank, then according to the space decomposition $H = \operatorname{rng} P \dotplus \operatorname{rng}(I-P)$,

$$X_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} : \ A \in \mathcal{B}(\operatorname{rng} P) \right\},$$

which implies that every element in X_1 is a sum of at most five idempotents in X_1 by [16, Theorem 1]. Since $\Phi(\mathbb{C}P_i) \subset \mathbb{C}\Phi(P_i)$ holds for rank-1 idempotents P_i by assumption, it turns out, in any case, we need only check $\Phi(Q) \in Y_1$ for idempotents Q in X_1. Clearly $Q(I-P) = (I-P)Q = 0$, which implies $\Phi(Q) \in Y_1$, since Φ preserves the orthogonality of idempotents. So $\Phi(X_1) \subset Y_1$. Similarly we can prove $\Phi(X_4) \subset Y_4$. Applying the surjectivity of Φ, we finish the proof of this step.

Step 3. There is an additive function $\tau : \mathbb{C} \to \mathbb{C}$ such that $\Phi(\lambda P) = \tau(\lambda)\Phi(P)$ holds for each rank-1 idempotent P.

Let $x \otimes y$ be an idempotent; then $\Phi(\lambda x \otimes y) \in \mathbb{C}\Phi(x \otimes y)$ by assumption. We can write $\Phi(\lambda x \otimes y) = \tau(\lambda)\Phi(x \otimes y)$. We claim that τ is independent of x and y. Indeed, we need to only show that τ does not depend on x; the remaining part can be proved in the same way. Let x, x', y be nonzero vectors and suppose that $\langle x, y\rangle = \langle x', y\rangle = 1$, $\langle x, x'\rangle = 0$. Let $\tau, \tau', \tau'' : \mathbb{C} \to \mathbb{C}$ be additive functions such that

$$\Phi(\lambda x \otimes y) = \tau(\lambda)\Phi(x \otimes y),$$

$$\Phi(\lambda x' \otimes y) = \tau'(\lambda)\Phi(x' \otimes y),$$

and

$$\Phi\left(\lambda(x+x')\otimes\frac{y}{2}\right)=\tau''(\lambda)\Phi\left((x+x')\otimes\frac{y}{2}\right).$$

Then we have

$$\begin{aligned}&\frac{1}{2}\tau''(\lambda)\Phi(x\otimes y)+\frac{1}{2}\tau''(\lambda)\Phi(x'\otimes y)\\&\quad=\frac{1}{2}\tau(\lambda)\Phi(x\otimes y)+\frac{1}{2}\tau'(\lambda)\Phi(x'\otimes y),\\&\big(\tau''(\lambda)-\tau(\lambda)\big)\Phi(x\otimes y)=\big(\tau'(\lambda)-\tau''(\lambda)\big)\Phi(x'\otimes y),\end{aligned}$$

and hence

$$\big(\tau'(\lambda)-\tau''(\lambda)\big)\Phi(x'\otimes y)\in\Phi(x\otimes y)\mathcal{B}(K)\Phi(x\otimes y).$$

It follows from Step 2 and the injectivity of Φ that $\tau'(\lambda)-\tau''(\lambda)\neq 0$ would imply $x'\otimes y\in(x\otimes y)\mathcal{B}(H)(x\otimes y)$, which is contrary to $\langle x,x'\rangle=0$. Thus $\tau'(\lambda)=\tau''(\lambda)$. Similarly, $\tau'(\lambda)=\tau(\lambda)$ and hence $\tau(\lambda)=\tau''(\lambda)$. If $\langle x,x'\rangle\neq 0$, one can take a third nonzero vector x'' which is orthogonal to x and x'. Then an analogous argument assures us that the claim of this step is true.

Step 4. $\Phi(\lambda x\otimes y)=\tau(\lambda)\Phi(x\otimes y)$ holds for arbitrary $x,y\in H$.

We will consider x,y in two cases.

Case I. $\langle x,y\rangle=0$. One can find $x'\in H$ such that $\langle x',y\rangle=1$. Thus

$$\begin{aligned}\Phi\big(\lambda(x-x')\otimes y+\lambda x'\otimes y\big)&=\tau(\lambda)\Phi\big((x-x')\otimes y\big)+\tau(\lambda)\Phi(x'\otimes y)\\&=\tau(\lambda)\Phi(x\otimes y).\end{aligned}$$

Case II. $\langle x,y\rangle\neq 0$. In this case we have

$$\Phi(x\otimes y)=\Phi\left(\langle x,y\rangle\frac{x}{\langle x,y\rangle}\otimes y\right)=\tau\big(\langle x,y\rangle\big)\Phi\left(\frac{x}{\langle x,y\rangle}\otimes y\right).$$

So

$$\begin{aligned}\Phi(\lambda x\otimes y)&=\Phi\left(\lambda\langle x,y\rangle\frac{x}{\langle x,y\rangle}\otimes y\right)\\&=\tau\big(\lambda\langle x,y\rangle\big)\Phi\left(\frac{x}{\langle x,y\rangle}\otimes y\right)=\frac{\tau(\lambda\langle x,y\rangle)}{\tau(\langle x,y\rangle)}\Phi(x\otimes y).\end{aligned}$$

Denote $\mu(\lambda)=\tau(\lambda\langle x,y\rangle)/\tau(\langle x,y\rangle)$. We need to show that $\tau(\lambda)=\mu(\lambda)$. Take $x'\in H$ such that $\langle x,x'\rangle=\langle x',y\rangle=0$ and suppose $\Phi(\lambda(x+x')\otimes y)=\mu'(\lambda)\Phi((x'+x)\otimes y)$. Then we have $\Phi(\lambda x'\otimes y)=\tau(\lambda)\Phi(x'\otimes y)$ by case I. A simple computation shows that

$$\begin{aligned}\mu'(\lambda)\Phi\big((x+x')\otimes y\big)=\Phi\big(\lambda(x+x')\otimes y\big)&=\Phi(\lambda x\otimes y)+\Phi(\lambda x'\otimes y)\\&=\mu(\lambda)\Phi(x\otimes y)+\tau(\lambda)\Phi(x'\otimes y),\end{aligned}$$

$$\big(\mu'(\lambda)-\mu(\lambda)\big)\Phi(x\otimes y)=\big(\tau(\lambda)-\mu'(\lambda)\big)\Phi(x'\otimes y),$$

and

$$\big(\mu'(\lambda)-\mu(\lambda)\big)\tau\big(\langle x,y\rangle\big)\Phi\left(\frac{x}{\langle x,y\rangle}\otimes y\right)=\big(\tau(\lambda)-\mu'(\lambda)\big)\Phi(x'\otimes y).$$

Using the result of Step 2 and the injectivity of Φ again, we get that if $\tau(\lambda)\neq\mu'(\lambda)$, then

$$x'\otimes y\in\left(\frac{x}{\langle x,y\rangle}\otimes y\right)\mathcal{B}(H)\left(\frac{x}{\langle x,y\rangle}\otimes y\right).$$

This is contrary to $\langle x,x'\rangle=0$. Thus $\tau(\lambda)=\mu(\lambda)$.

Step 5. $\tau(\lambda)\equiv\lambda$ or $\tau(\lambda)\equiv\bar{\lambda}$.

We first show that τ is a ring homomorphism. Indeed,

$$\tau(\lambda)\tau(\mu)\Phi(x\otimes y)=\tau(\lambda)\Phi(\mu x\otimes y)=\Phi(\lambda\mu x\otimes y)=\tau(\lambda\mu)\Phi(x\otimes y)$$

and this implies that τ is multiplicative. We assert that τ is continuous, too. If τ is not continuous, there is a bounded sequence $\{\lambda_n\}\subset\mathbb{C}$ such that $|\tau(\lambda_n)|\to\infty$. Let $\{P_n\}$ be a sequence of mutually orthogonal rank-1 projections. Consider the operator $A=\sum_n\lambda_nP_n\in\mathcal{B}(H)$. For any $n_0\in\mathbb{N}$, let $x\in\operatorname{rng}\Phi(P_{n_0})$ be a unit vector. Then, by Step 2, we see that

$$\begin{aligned}\|\Phi(A)\|\geqslant\|\Phi(A)x\|&=\left\|\Phi\left(\sum_n\lambda_nP_n\right)\Phi(P_{n_0})x\right\|\\&=\left\|\Phi(\lambda_{n_0}P_{n_0})\Phi(P_{n_0})x+\Phi\left(\sum_{n\neq n_0}\lambda_nP_n\right)\Phi(P_{n_0})x\right\|\\&=\big|\tau(\lambda_{n_0})\big|.\end{aligned}$$

This means that the operator $\Phi(A)$ is not bounded, which is a contradiction. Therefore, τ is continuous. Since every nontrivial continuous ring endomorphism of $\mathbb{C}$ is either the identity or the conjugation (see [17, p. 356, Lemma 1]), the assertion is true.

Step 6. Both Φ and Φ^{-1} are rank preserving.

We need only to check the assertion for Φ since Φ^{-1} has the same properties as Φ. It is clear from Steps 1 and 2 that Φ preserves rank-1 idempotents. A similar argument as that in the proof of [7] shows that Φ is rank-1 preserving. Since Φ is surjective and linear or conjugate-linear on $\mathcal{F}(H)$, it is not difficult to see that Φ preserves the rank of operators.

Step 7. There exists a linear or conjugate-linear bijective operator A from H onto K such that $\Phi(T)A=AT$ for all $T\in\mathcal{F}(H)$ or $\Phi(T)A=AT^*$ holds for every $T\in\mathcal{F}(H)$.

Let $\Psi=\Phi|_{\mathcal{F}(H)}:\mathcal{F}(H)\to\mathcal{F}(K)$. Then Ψ is bijective and either linear or conjugate-linear.

We now show that Ψ is a Jordan homomorphism on $\mathcal{F}(H)$. Let $S \in \mathcal{F}(H)$ be self-adjoint. Then $S = \sum_{k=1}^{n} \lambda_k P_k$ with $\{\lambda_k\} \subset \mathbb{R}$ and $\{P_k\}$ a set of pairwise orthogonal rank-1 projections. Thus we have

$$\Psi(S)^2 = \left(\sum_{k=1}^{n} \lambda_k \Psi(P_k)\right)^2 = \sum_{k=1}^{n} \lambda_k^2 \Psi(P_k) = \Psi\left(\sum_{k=1}^{n} \bigl(\lambda_k^2 P_k\bigr)\right) = \Psi(S^2).$$

Replacing S by $S + T$, we obtain that

$$\Psi(ST + TS) = \Psi(S)\Psi(T) + \Psi(T)\Psi(S)$$

holds for every pair of self-adjoint operators $S, T \in \mathcal{F}(H)$. Since every operator in $\mathcal{F}(H)$ can be written as $S + iT$ with S and T self-adjoint, and since

$$\begin{aligned}\Psi(S + iT)^2 &= \bigl(\Psi(S) + \tau(i)\Psi(T)\bigr)^2\\ &= \Psi(S^2) + \tau(i)^2\Psi(T)^2 + \tau(i)\bigl(\Psi(S)\Psi(T) + \Psi(T)\Psi(S)\bigr)\\ &= \Psi(S^2) - \Psi(T^2) + \tau(i)\bigl(\Psi(S)\Psi(T) + \Psi(T)\Psi(S)\bigr)\\ &= \Psi\bigl(S^2 - T^2 + i(ST + TS)\bigr) = \Psi\bigl((S + iT)^2\bigr),\end{aligned}$$

we see that $\Psi : \mathcal{F}(H) \to \mathcal{F}(K)$ is a Jordan homomorphism. By [18, Theorem 8], $\Psi = \varphi + \psi$, where φ is a ring homomorphism and ψ is a ring anti-homomorphism from $\mathcal{F}(H)$ onto $\mathcal{F}(K)$. Since $\mathcal{F}(K)$ is prime, $\Psi = \varphi$ or $\Psi = \psi$.

Now, we turn to determine A. Take $x_0, y_0, \in H$ and $z \in K$ such that $\Psi(x_0 \otimes y_0)z \neq 0$.

If $\Psi = \varphi$, define $Ax = \Psi(x \otimes y_0)z$, for each $x \in H$. Then the linearity and conjugate-linearity of Ψ implies the same of A, respectively. And clearly, $ATx = \Psi(Tx \otimes y_0)z = \Psi(T)\Psi(x \otimes y_0)z = \Psi(T)Ax$. So $AT = \Psi(T)A$. To see A is injective, suppose that there is a nonzero vector x such that $Ax = 0$. Then $ATx = 0$ for every $T \in \mathcal{F}(H)$, which implies $A = 0$; a contradiction. A is also surjective because Ψ is.

If $\Psi = \psi$, define $A : H \to K$ by $Ay = \Phi(x_0 \otimes y)z$. In the same way as above, one can verify $\Psi(T)A = AT^*$, A is linear or conjugate-linear bijective operator according to whether Ψ is.

Step 8. For every projection P, $\Phi(\lambda P) = \tau(\lambda)\Phi(P)$ holds for all $\lambda \in \mathbb{C}$.

It is clear that there exists a set $\{P_\alpha\}_{\alpha\in\Lambda}$ of mutually orthogonal rank-1 projections such that $P = \sum_{\alpha\in\Lambda} P_\alpha$, where Λ is a index set. We first show that $\Phi(P) = \sum_\alpha \Phi(P_\alpha)$. Note that

$$\Phi(P)\left(\sum_\alpha \Phi(P_\alpha)\right) = \left(\sum_\alpha \Phi(P_\alpha)\right)\Phi(P) = \sum_\alpha \Phi(P_\alpha).$$

Suppose $G = \Phi(P) - \sum_\alpha \Phi(P_\alpha)$. By Step 2, there is $R \in P\mathcal{B}(H)P$ such that $\Phi(R) = G$. Since $G\Phi(P_\alpha) = \Phi(P_\alpha)G = 0$ for each $\alpha \in \Lambda$ and Φ is injective, we have $R \in (I - P_\alpha)\mathcal{B}(H)(I - P_\alpha)$. Thus $R = RP = R(\sum_\alpha P_\alpha) = 0$. This yields $G = 0$.

Now for any $x \in \operatorname{rng} \Phi(P_{\alpha_0})$, we have

$$\begin{aligned}\Phi(\lambda P)x &= \left(\Phi(\lambda P_{\alpha_0}) + \sum_{\alpha \neq \alpha_0} \Phi(\lambda P_\alpha)\right)\Phi(P_{\alpha_0})x \\ &= \Phi(\lambda P_{\alpha_0})\Phi(P_{\alpha_0})x + \left(\sum_{\alpha \neq \alpha_0} \Phi(\lambda P_\alpha)\right)\Phi(P_{\alpha_0})x = \tau(\lambda)x.\end{aligned}$$

Thus, for every $x \in \operatorname{rng} \Phi(P)$, we have $\Phi(\lambda P)x = \tau(\lambda)\Phi(P)x$. Since $\lambda P \in P\mathcal{B}(H)P$, so, by Step 2, $\Phi(\lambda P) \in \Phi(P)\mathcal{B}(K)\Phi(P)$ and hence, for any $x \in \operatorname{rng}(I - \Phi(P))$,

$$\Phi(\lambda P)x = \Phi(\lambda P)\big(I - \Phi(P)\big)x = 0$$

and

$$\tau(\lambda)\Phi(P)x = \tau(\lambda)\Phi(P)\big(I - \Phi(P)\big)x = 0.$$

Now it is clear that $\Phi(\lambda P) = \tau(\lambda)\Phi(P)$.

Step 9. A is bounded and $\Phi(T) = ATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = AT^*A^{-1}$ for all $T \in \mathcal{B}(H)$.

First, we show that $\Phi(\lambda P)A = A(\lambda P)$ (or $\Phi(\lambda P)A = A(\bar{\lambda} P)$) holds for every projection P. Write P in the form as that in Step 8. Then

$$\Phi(\lambda P) = \tau(\lambda)\Phi(P) = \tau(\lambda)\sum_\alpha \Phi(P_\alpha) = \sum_\alpha \Phi(\lambda P_\alpha)$$

and

$$\begin{aligned}\Phi(\lambda P)A &= \left(\sum_\alpha \Phi(\lambda P_\alpha)\right)A = \sum_\alpha \big(\Phi(\lambda P_\alpha)A\big) \\ &= \sum_\alpha A(\lambda P_\alpha) = A\sum_\alpha \lambda P_\alpha = A(\lambda P)\end{aligned}$$

(or $\Phi(\lambda P)A = \sum_\alpha A(\lambda P_\alpha)^* = A(\bar{\lambda} P)$). Since by [16] every element of $\mathcal{B}(H)$ can be written as a linear combination of a finite number of projections, $\Phi(T)A = AT$ (or $\Phi(T)A = AT^*$) hold for all $T \in \mathcal{B}(H)$. From the bijectivity of A we get $\Phi(T) = ATA^{-1}$ (or $\Phi(T) = AT^*A^{-1}$). Now by the closed graph Theorem one can get easily that A is bounded, completing the proof. □

Comparing this result with the linear ones (Theorem 2.3), we conjecture that the following assertion is true, but we are not able to prove it here.

Conjecture 3.2. *Let H and K be infinite-dimensional Hilbert spaces and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a continuous surjective additive map. Then Φ preserves square zero operators if and only if there exists a nonzero scalar c and a bounded invertible linear or conjugate-linear operator A from H onto K such that either $\Phi(T) = cATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = cAT^*A^{-1}$ for all $T \in \mathcal{B}(H)$.*

In order to characterize additive maps preserving nilpotents with higher nilindex in both directions, we need two lemmas which can be obtained from [8, Lemmas 2.2 and 2.3].

Lemma 3.3. *Let $k > 2$ be a positive integer, H a (finite- or infinite-dimensional) Hilbert space, and let $T \in \mathcal{N}^k(H)$ be a nonzero operator. Then the following conditions are equivalent*:

(i) $T \in \mathcal{N}_1(H)$.
(ii) *For every $S \in \mathcal{N}^k(H)$ satisfying $T + S \notin \mathcal{N}^k(H)$ we have $2T + S \notin \mathcal{N}^k(H)$.*

Lemma 3.4. *Let H be an infinite-dimensional Hilbert space, k an integer not smaller than 3, and let $T \in \mathcal{B}(H)$ be a nonzero square-zero operator. Assume also that T is not a rank-1 operator. Let S be any operator from $\mathcal{B}(H)$. Suppose that for every finite rank nilpotent operator $R \in \mathcal{B}(H)$ the operator $T + R \in \mathcal{N}^k(H)$ if and only if $R + S \in \mathcal{N}^k(H)$. Then $T = S$.*

Let τ be a ring automorphism of a number field $\mathbb{F}$. Recall that a transformation A between linear spaces over $\mathbb{F}$ is said to be τ-quasilinear if $A(x + \lambda y) = Ax + \tau(\lambda)Ay$ for any vectors x, y and any $\lambda \in \mathbb{F}$. The following lemma is proved in [19].

Lemma 3.5. *Let X, Y be Banach spaces over $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with $\dim X \geqslant 4$ and let Ω_X (Ω_Y) be a subspace of $\mathcal{B}(X)$ ($\mathcal{B}(Y)$) which contains all nilpotent operators of $\mathcal{B}(X)$ ($\mathcal{B}(Y)$). Suppose that $\Phi : \Omega_X \to \Omega_Y$ is a bijective additive map which preserves rank-1 nilpotent operators in both directions. Then there is a nonzero number $c \in \mathbb{F}$ and a ring automorphism τ of $\mathbb{F}$ such that either*

(i) *there exists a bijective τ-quasilinear transformation $A : X \to Y$ such that $\Phi(x \otimes f) = cA(x \otimes f)A^{-1}$ for every $x \in X$, $f \in X'$ with $f(x) = 0$, or*
(ii) *there exists a bijective τ-quasilinear transformation $A : X' \to Y$ such that $\Phi(x \otimes f) = cA(x \otimes f)^* A^{-1}$ for every $x \in X$, $f \in X'$ with $f(x) = 0$. In this case, X and Y are reflexive.*

In particular, if X is infinite-dimensional, then A is in fact a bounded linear or conjugate-linear operator.

Theorem 3.6. *Let H and K be infinite-dimensional Hilbert spaces, k an integer not smaller than 3, and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a surjective additive map. Then Φ preserves k-nilpotent operators in both directions if and only if there exists a scalar c and a bounded invertible linear or conjugate-linear operator A from H onto K such that either $\Phi(T) = cATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = cAT^*A^{-1}$ for all $T \in \mathcal{B}(H)$.*

Proof. First we show that Φ is injective. If $\Phi(T) = 0$ for some nonzero $T \in \mathcal{B}(H)$, then $T^k = 0$ and it is easy to find an operator $S \in \mathcal{B}(H)$ such that $T + S \notin \mathcal{N}^k(H)$. It follows that $\Phi(S) = \Phi(T + S) \notin \mathcal{N}^k(K)$ which is a contradiction.

Next we claim that Φ preserves rank-1 nilpotent operators in both directions. Indeed, we need only to show that Φ preserves rank-1 nilpotent operators. To do this let $T \in \mathcal{N}_1(H)$ be arbitrary; we shall show that $\Phi(T) \in \mathcal{N}_1(K)$. For every $S \in \mathcal{N}^k(K)$, take $R \in \mathcal{N}^k(H)$ satisfying $\Phi(R) = S$. If $S + \Phi(T) \notin \mathcal{N}^k(K)$, then $T + R \notin \mathcal{N}^k(H)$. Thus by Lemma 3.3, $2T + R \notin \mathcal{N}^k(H)$. Consequently, $2\Phi(T) + S \notin \mathcal{N}^k(K)$. Using Lemma 3.3 again, we get $\Phi(T) \in \mathcal{N}_1(K)$. By Lemma 3.5, there is a nonzero number $c \in \mathbb{C}$ and an invertible bounded linear or conjugate-linear operator $A : H \to K$ such that either

(i) $\Phi(x \otimes f) = cA(x \otimes f)A^{-1}$ for any $x, f \in H$ with $\langle x, f\rangle = 0$, or
(ii) $\Phi(x \otimes f) = cA(x \otimes f)^* A^{-1}$ for any $x, f \in H$ with $\langle x, f\rangle = 0$.

It is well known that every finite rank nilpotent operator is a sum of some rank-1 nilpotent operators. Let $T \in \mathcal{F}(H) \cap \mathcal{N}(H)$ be arbitrary. Assume that $T = \sum_{i=1}^n x_i \otimes f_i$ for some $x_i, f_i \in H$ with $\langle x_i, f_i\rangle = 0$ $(i = 1, 2, \ldots, n)$. Then in the case (i),

$$\begin{aligned}\Phi(T) &= \sum_{i=1}^n \Phi(x_i \otimes f_i) = \sum_{i=1}^n cA(x_i \otimes f_i)A^{-1} \\ &= c\sum_{i=1}^n A(x_i \otimes f_i)A^{-1} = cATA^{-1}.\end{aligned}$$

Let us define a new mapping $\Psi : B(H) \to B(H)$ by $\Psi(T) = c^{-1}A^{-1}\Phi(T)A$. Obviously Ψ is a bijective additive map preserving k-nilpotent operators in both directions. Moreover, $\Psi(N) = N$ for every finite rank nilpotent operator N. Let T be an arbitrary square zero operator of infinite rank. For any $N \in \mathcal{F}(H) \cap \mathcal{N}(H)$, if $T + N \in \mathcal{N}^k(H)$, then $\Psi(T) + N = \Psi(T) + \Psi(N) = \Psi(T + N) \in \mathcal{N}^k(H)$. By Lemma 3.4, we get $T = \Psi(T)$. Since every operator in $\mathcal{B}(H)$ is a sum of at most 5 square zero operators, we conclude that $\Psi(T) = T$ for all T, and consequently, $\Phi(T) = cATA^{-1}$ for all $T \in B(H)$. If the case (ii) occurs, then similarly $\Phi(T) = cAT^*A^{-1}$ for every $T \in \mathcal{F}(H) \cap \mathcal{N}(H)$. Let Ψ be a map defined by $\Psi(T) = (c^{-1}A^{-1}\Phi(T)A)^*$; then, again, Ψ is a bijective additive map preserving nilpotent operators with nilindex not greater than k in both directions, and hence, $\Psi(T) = T$ for all T. It follows immediately that $\Phi(T) = cAT^*A^{-1}$ for every $T \in B(H)$. $\square$

From Theorem 3.6 and [19, Theorem 2.3] one can easily get the following corollary.

Corollary 3.7. *Let H and K be infinite-dimensional Hilbert spaces, k an integer not smaller than* 3, *and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a surjective additive map. The following conditions are equivalent*:

(i) *Φ preserves k-nilpotent operators in both directions.*
(ii) *Φ preserves nilpotent operators in both directions.*

Next, we turn to give our main result. Let $p(t) = (t - t_1)\dots(t - t_k)$ be a complex polynomial with $\deg(p) = k \geqslant 2$; here $t_1, \dots, t_k$ are possibly repeated complex numbers. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a surjective additive map that preserves operators annihilated by $p(t)$ in both directions. First we will show that Φ preserves k-nilpotent operators in both directions. Let $N \in \mathcal{B}(H)$ be a nilpotent with nilindex $r \leqslant k$. Then there exists a direct sum decomposition of H into closed subspaces $H = H_1 \oplus H_2 \oplus \dots \oplus H_r$ such that

$$N = \begin{bmatrix} 0 & N_{12} & N_{13} & \dots & N_{1r} \\ 0 & 0 & N_{23} & \dots & N_{2r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & N_{(r-1)r} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

with respect to this decomposition (see [8]). If

$$A = \begin{bmatrix} t_1 I & 0 & \dots & 0 \\ 0 & t_2 I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_r I \end{bmatrix}$$

then $p(A + \alpha N) = 0$ for every rational number α. It follows that $p(\Phi(A) + \alpha\Phi(N)) = 0$ for every rational number α. All rational numbers are roots of the polynomial $q(t) = p(\Phi(A) + t\Phi(N))$. Thus all coefficients of this operator polynomial must be zero. In particular, the coefficient of t^k must be zero, and hence $\Phi(N)^k = 0$. Conversely, assume that $\Phi(M)^k = 0$ for some operator $M \in \mathcal{B}(H)$. As above we can find $C \in \mathcal{B}(K)$ such that $p(C + \alpha\Phi(M)) = 0$ for all $\alpha \in \mathbb{C}$. The surjectivity of Φ yields the existence of $D \in \mathcal{B}(H)$ such that $\Phi(D) = C$. It follows that $p(D + \alpha M) = 0$ for every rational number α, and consequently, $M^k = 0$.

Now we are at a position to state our main result in this section.

Theorem 3.8. *Let H and K be infinite-dimensional complex Hilbert spaces, and let $p(t)$ be a complex polynomial with* $\deg(p) \geqslant 2$. *Assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a surjective additive map satisfying $\Phi(\mathbb{C}P) \subset \mathbb{C}\Phi(P)$ for every rank*-1 *idempotent P. Then Φ preserves operators annihilated by $p(t)$ in both directions if and only if there exists a complex number $c \in \mathcal{G}(p)$ and an invertible bounded linear or conjugate-linear operator $A : H \to K$ such that either $\Phi(T) =$*

$cATA^{-1}$ for all $T \in \mathcal{B}(H)$ or $\Phi(T) = cAT^*A^{-1}$ for all $T \in \mathcal{B}(H)$. Moreover, if $\deg(p) > 2$, the condition that $\Phi(\mathbb{C}P) \subset \mathbb{C}\Phi(P)$ for every rank-1 idempotent P can be removed.

Proof. It is an immediate consequence of the above observation and Theorems 3.1 and 3.6. □

We remark that Φ takes the form $\Phi(T) = cAT^*A^{-1}$ for all $T \in \mathcal{B}(H)$ if and only if there is an invertible bounded conjugate-linear or linear operator B such that $\Phi(T) = cBT^{\mathrm{tr}}B^{-1}$ for all $T \in \mathcal{B}(H)$, where T^{tr} denotes the transpose of T relative to a fixed but arbitrary orthonormal base of H.

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